Abelian surfaces, Siegel modular forms, and the Paramodularity Conjecture

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Motivation: Modularity of elliptic curves

$$\left\{ \begin{array}{c} \text{Elliptic curves} \\ \text{over } \mathbb{Q} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Classical modular} \\ \text{forms with rational} \\ \text{Hecke eigenvalues} \end{array} \right\}$$

The correspondence ensures equality of L-functions.

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- State the Paramodularity Conjecture
- Explore the first known case of the Paramodular conjecture

Elliptic curves

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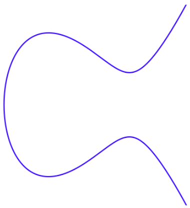
$$Y^2 = X^3 + AX + B$$

with $A, B \in \mathbb{Q}$, $\Delta = 4A^3 + 27B^2 \neq 0$.



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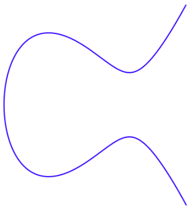

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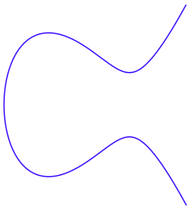

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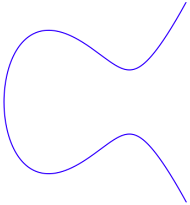
- We can assume $A, B, \Delta \in \mathbb{Z}$.



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- The L-function of E is the function

$$L(E,s) = \prod_{p \mid \Delta}(1-a_p p^{-s})^{-1} \prod_{p \nmid \Delta}(1-a_p p^{-s}+p^{1-2s})^{-1}$$

which is holomorphic for $\mathrm{Re}(s) > \frac{3}{2}$.

Modular forms

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- We consider the group of matrices $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \right\}$.
- A holomorphic $\mathrm{f} : \mathcal{H} \to \mathbb{C}$ is a modular form of level N if

$$\mathrm{f}(\gamma \cdot z) = (cz+d)^2\, \mathrm{f}(z)$$

for all $\gamma \in \Gamma_0(N)$, and $\mathrm{f}(z)$ satisfies some boundedness condition.

Cusp forms

- Since $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$, $\mathsf{f}(Tz) = \mathsf{f}(z+1) = \mathsf{f}(z)$. Hence modular forms have a Fourier expansion

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- Modular forms: $M_2(\Gamma_0(N))$.
- Cusp forms: $S_2(\Gamma_0(N))$ (both are $\mathbb{C}$-vector spaces).

Hecke operators and L-functions

- Given p prime, the Hecke operator $T_p : M_2(\Gamma_0(N)) \to M_2(\Gamma_0(N))$ preserves the space of cusp forms.

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- The L-function of f is given by

$$\begin{aligned} L(\mathrm{f}, s) &= \prod_{p \mid N} (1 - a_p(f) p^{-s})^{-1} \prod_{p \nmid N} (1 - a_p(f) p^{-s} + p^{1-2s})^{-1} \\ &= \sum_{n \geq 1} \frac{a_n(f)}{n^s}. \end{aligned}$$

Modularity of Elliptic curves

Modularity Theorem

Let $E_{/\mathbb{Q}}$ be an elliptic curve of conductor N. Then there is an eigenform $f \in S_2(\Gamma_0(N))$ such that

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$$\operatorname{rank}_{\mathbb{Z}} E(\mathbb{Q}) = \operatorname{ord}_{s=1} L(E,s)$$

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  - $E \rightsquigarrow f_E \in S_2(\Gamma_0(2)) = \{0\}$ (!!)

Abelian varieties

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  for $d = \dim A$ and some discrete subgroup $\Lambda \subset \mathbb{C}^d$ of rank $2d$.
- Given a Riemann surface C, its *Jacobian* is the quotient

$$\mathrm{Jac}(C) = \Omega^1(C)^\vee/H_1(C, \mathbb{Z}),$$

  where $\Omega^1(C)$ is the $\mathbb{C}$-vector space of holomorphic differentials, and $H_1(C, \mathbb{Z})$ is the integral homology on C.

L-functions

- For each prime p, the pth Euler factor of $A_{/\mathbb{Q}}$ is a polynomial

$$L_p(A, T) \in 1 + T\mathbb{C}[T].$$

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- We use the Euler factors to define L-function of A,

$$L(A, s) = \prod_p L_p(A, p^{-s})^{-1}.$$

  It converges in a right-hand plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \frac{3}{2}\}$.

$X_0(N)$ and $J_0(N)$

- The quotient $\Gamma_0(N) \setminus \mathcal{H}$ can be compactified to form a compact Riemann surface,

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- $X_0(N)$ has an algebraic model over $\mathbb{Q}$, the *modular curve of level N*.
- The Jacobian of $X_0(N)$ is denoted by $J_0(N)$. We have an isomorphism
$$\Omega^1(X_0(N)) \cong S_2(\Gamma_0(N)),$$
which gives us the expression
$$J_0(N) \cong S_2(\Gamma_0(N))^\vee / H_1(X_0(N), \mathbb{Z}).$$

Eichler-Shimura theorem

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- Let $K_f = \mathbb{Q}(\{a_n\})$ be the number field generated by its coefficients.
- Then there exists an abelian variety A_f of dimension $[K_f : \mathbb{Q}]$, such that
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- In particular, if f has integer Fourier coefficients then A_f is an elliptic curve.

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- Let $\mathbb{T}_{\mathbb{Z}}$ be the $\mathbb{Z}$-algebra generated by the operators T_p, acting on $S_2(\Gamma_0(N))$.

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- Given an eigenform $f \in S_2(\Gamma_0(N))$, we have a homomorphism

$$\lambda_f : \mathbb{T}_{\mathbb{Z}} \to \mathbb{C}$$

such that $T_p f = \lambda_f(T_p) f$.

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- Let $I_f = \ker \lambda_f$. Then the quotient

$$A_f := J_0(N)/I_f J_0(N)$$

  is an abelian variety.

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  is an abelian variety.
- Hard part: prove equality of L-functions.

$\square$

Modularity of GL_2-type abelian varieties

- Given an eigenform f, the endomorphism ring of the variety A_f satisfies
$$\mathrm{End}(A_f) \otimes \mathbb{Q} \cong K_f.$$
- An abelian variety $A_{/\mathbb{Q}}$ is said to be of GL_2-type if $\mathrm{End}(A) \otimes \mathbb{Q}$ contains a number field of degree $\dim A$.

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Theorem (Ribet)

Let $A_{/\mathbb{Q}}$ be an abelian variety of GL_2-type of conductor N with endomorphism algebra K. There exists a classical modular eigenform $f \in S_2(\Gamma_1(N))$ such that

$$L(A, s) = \prod_{\sigma : K \hookrightarrow \mathbb{C}} L(f^\sigma, s).$$

Siegel upper half space, symplectic action

Let's build a higher-dimensional analogy of modular forms:

- First, the space: the Siegel upper-half space is

$$\mathcal{H}_2 = \{Z \in \mathrm{Mat}_{2\times 2}^{sym}(\mathbb{C}) \mid \mathrm{Im}\, Z > 0\}.$$

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- The symplectic group is

$$\mathrm{Sp}_4(\mathbb{R}) = \{M \in \mathrm{GL}_4(\mathbb{R}) \mid M^T \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}\}.$$

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- $\mathrm{Sp}_4(\mathbb{R})$ acts on $\mathcal{H}_2$ as

  $$Z \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

Siegel Paramodular Forms

Let $N \in \mathbb{Z}_{\geq 1}$. Our substitute for $\Gamma_0(N)$ is the level N paramodular group

$$K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Q})$$

where $* \in \mathbb{Z}$.

Siegel Paramodular Forms

Let $N \in \mathbb{Z}_{\geq 1}$. Our substitute for $\Gamma_0(N)$ is the level N paramodular group

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where $* \in \mathbb{Z}$.
A Siegel paramodular form of level N and weight 2 is a holomorphic $\mathrm{f} : \mathcal{H}_2 \to \mathbb{C}$ satisfying

- $\mathrm{f}(MZ) = \det(CZ + D)^2\, \mathrm{f}(Z)$ for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K(N)$.
- $\mathrm{f}(Z)$ satisfies a boundedness condition.

Koecher principle

Theorem

A paramodular form $f \in M_2(K(N))$ has a Fourier expansion of the form

$$f\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = \sum_{T \geq 0} a(T; f) e^{2\pi i(n\tau + rz + Nm\omega)},$$

where T runs over positive semidefinite matrices of the form

$$T = \begin{pmatrix} n & r/2 \\ r/2 & Nm \end{pmatrix}, \ n, r, m \in \mathbb{Z}.$$

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- There is a notion of cusp forms, $S_2(K(N))$.
- They satisfy $a(T; f) = 0$ for $\det T = 0$.

Hecke operators and L-functions

- One defines Hecke operators

$$T(p) : M_2(K(N)) \to M_2(K(N)),$$
$$T_1(p^2) : M_2(K(N)) \to M_2(K(N))$$

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- If f is a simultaneous eigenform for all $T(p)$, $T_1(p^2)$, its eigenvalues let us define *spinor Euler factors*

$$Q_p(f, T) \in 1 + T\mathbb{C}[T], \quad \deg Q_p = 4$$

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- We package the Euler factors into the L-function of f,

$$L(\mathrm{f}, s) = \prod_p Q_p(1, p^{-s})^{-1}.$$

Gritsenko lift

- Jacobi forms: holomorphic functions

$$\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$$

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Theorem (Gritsenko)

There is a lift

$$\mathrm{Grit} : J_{2,N}^{cusp} \to S_2(K(N)).$$

The paramodular form $\mathrm{Grit}(\phi)$ is given explicitly in terms of the Fourier expansion of ϕ.

Paramodularity Conjecture

Let $A_{/\mathbb{Q}}$ be an abelian surface of conductor N with $\mathrm{End}(A) = \mathbb{Z}$. There exists a Siegel paramodular eigenform $f_A \in S_2(K(N))$ which is not a Gritsenko lift such that

$$L(A, s) = L(f_A, s).$$

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Conversely, given a nonlift Siegel paramodular eigenform $f \in S_2(K(N))$ of squarefree level N, there is an abelian surface $A_{f/\mathbb{Q}}$ of conductor N with $\mathrm{End}(A) = \mathbb{Z}$ and such that

$$L(f, s) = L(A_f, s).$$

Example: the nonlift of level 277

Theorem (Poor-Yuen)

The subspace of Gritsenko lifts of $S_2(K(277))$ has dimension 10, whereas $\dim S_2(K(277)) = 11$. The form f_{277} is a Hecke eigenform with rational eigenvalues which is not a Gritsenko lift, given as a degree-2 rational function of Gritsenko lifts $G_1, \ldots, G_{10} \in S_2(K(277))$.

Example: the nonlift of level 277

$$
\begin{aligned}
f_{277} = (&-14G_1^2 - 20G_8G_2 + 11G_9G_2 + 6G_2^2 - 30G_7G_{10} + 15G_9G_{10} \\
&+ 15G_{10}G_1 - 30G_{10}G_2 - 30G_{10}G_3 + 5G_4G_5 + 6G_4G_6 + 17G_4G_7 \\
&- 3G_4G_8 - 5G_4G_9 - 5G_5G_6 + 20G_5G_7 - 5G_5G_8 - 10G_5G_9 - 3G_6^2 \\
&+ 13G_6G_7 + 3G_6G_8 - 10G_6G_9 - 22G_7^2 + G_7G_8 + 15G_7G_9 + 6G_8^2 \\
&- 4G_8G_9 - 2G_9^2 + 20G_1G_2 - 28G_3G_2 + 23G_4G_2 + 7G_6G_2 \\
&- 31G_7G_2 + 15G_5G_2 + 45G_1G_3 - 10G_1G_5 - 2G_1G_4 - 13G_1G_6 \\
&- 7G_1G_8 + 39G_1G_7 - 16G_1G_9 - 34G_3^2 + 8G_3G_4 + 20G_3G_5 \\
&+22G_3G_6 + 10G_3G_8 + 21G_3G_9 - 56G_3G_7 - 3G_4^2) \; / \\
&(-G_4 + G_6 + 2G_7 + G_8 - G_9 + 2G_3 - 3G_2 - G_1)
\end{aligned}
$$

Paramodularity for $N = 277$

Theorem (Brumer-Pacetti-Poor-Tornaría-Voight-Yuen)

Let C be the curve over $\mathbb{Q}$ defined by

$$C : y^2 + (x^3 + x^2 + x + 1)y = -x^2 - x.$$

Let A be the Jacobian of C, which has $\mathrm{End}(A) = \mathbb{Z}$ and $N = 277$.

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$$L_p(A, T) = Q_p(f_{277}, T).$$

In particular $L(A, s) = L(f_{277}, s)$ and A is paramodular.

Strategy of proof

- Generalized Faltings-Serre method to check

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- The method reduces the check to finitely many primes.
- Write

$$\begin{cases} L_p(A, T) & = 1 - a_p(A)T + \cdots + p^2 T^4 \\ Q_p(f_{277}, T) & = 1 - a_p(f_{277})T + \cdots + p^2 T^4. \end{cases}$$

  It is enough to compute $a_p(A)$ and $a_p(f_{277})$ for all primes

$$p \in \{2, 3, 5, \ldots, 41, 43\}.$$

Computation of $a_p(A)$: point counting

- We have $A = \mathrm{Jac}(C)$, with

$$C : y^2 + (x^3 + x^2 + x + 1)y = -x^2 - x.$$

- If $\tilde{C}_{/\mathbb{F}_p}$ is the reduction of C modulo p, then

$$a_p(A) = p + 1 - \#\tilde{C}(\mathbb{F}_p).$$

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- This can be done e.g. with Sage:

```
SageMath version 9.2, Release Date: 2020-10-24
Using Python 3.8.5. Type "help()" for help.

sage: R.<x> = QQ[]
sage: C = HyperellipticCurve(-x^2 - x, x^3 + x^2 + x + 1)
sage: C.change_ring(GF(2)).count_points()
[5]
sage: C.change_ring(GF(3)).count_points()
[5]
sage: C.change_ring(GF(5)).count_points()
[7]
sage:
```

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- To compute the previous expression for f_{277}, we need lots of Fourier coefficients of each individual Gritsenko lift G_i.
- Improvement: specialization.

Computation of Hecke eigenvalues

- Let $s \in \mathrm{Mat}^{sym}_{2\times 2}(\mathbb{Z})$ be positive definite, then

$$\begin{aligned} \phi_s : \mathcal{H}_1 &\to \mathcal{H}_2 \\ \tau &\mapsto s\tau \end{aligned}$$

yields a map $\phi_s^* : S_2(K(N)) \to S_2(\Gamma_0(\det(s)N))$.

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- The specialization morphism ϕ_s^* is a ring homomorphism, so we can specialize each lift individually

$$\phi_s^* f_{277} = Q(\phi_s^* G_1, \ldots, \phi_s^* G_{10}).$$

  This (one-variable) series is then compared with

$$\phi_s^*(T(p) f_{277}) = Q(\phi_s^*(T(p) G_1), \ldots, \phi_s^*(T(p) G_{10})).$$

Computation of Hecke eigenvalues

- The remaining task is to compute $G_i = \mathrm{Grit}(\Xi_i)$.
- Each Ξ_i is a *theta block*, computed by multiplying *two-variable Laurent series*.

$$\Xi_i \quad \rightsquigarrow \quad \mathrm{Grit}(\Xi_i) \quad \rightsquigarrow \quad \begin{cases} \phi_s^*(\mathrm{Grit}(\Xi_i)) \\ \phi_s^*(T(p)\,\mathrm{Grit}(\Xi_i)). \end{cases}$$

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- With the data from the LMFDB, we are able to compute $a_p(f_{277})$ up to $p \leq 23$.
- With the specialization method, we get $a_p(f_{277})$ for $p = 2, 3, 5$.
- Further a_p can be computed by increasing precision / computational resources.

Comparison of a_p's

```
enric@MacBookPro siegel-paramodular-forms % sage specialization.sage
N = 277
det(2T0) = 3, a(T0;f) = -3
-3*q^3 + O(q^4)

p = 2
6*q^3 + O(q^4)
a_p(f) = -2
a_p(C) = -2

p = 3
3*q^3 + O(q^4)
a_p(f) = -1
a_p(C) = -1

p = 5
3*q^3 + O(q^4)
a_p(f) = -1
a_p(C) = -1

p = 7
-3*q^3 + O(q^4)
a_p(f) = 1
a_p(C) = 1
```

Abelian surfaces, Siegel modular forms, and the Paramodularity Conjecture

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Master in Advanced Mathematics – Universitat de Barcelona

July 15, 2021