Let's twist again: the problem with fake abelian surfaces

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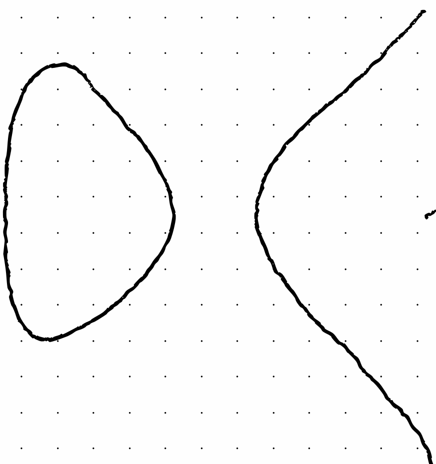
1. Introduction



$$\rightsquigarrow \quad L(s) \quad \rightsquigarrow \quad f(z) = \sum_{n \geq 0} a_n q^n$$

1.1. Modularity

An elliptic curve is defined by an equation

$$E: y^2 = x^3 + Ax + B.$$

Every elliptic curve has an attached meromorphic function called its L-function:

$$L(E,s) = \prod_{p \mid N} (1 - a_p p^{-s})^{-1} \prod_{p \nmid N} (1 - a_p p^{-s} + p^{1-2s})^{-1}.$$

$s \in \mathbb{C}$

The values a_p are computed by counting points on E modulo p.

A **modular form** is a holomorphic function

$$f : \{z \in \mathbb{C},\ \mathrm{Im}(z) > 0\} \longrightarrow \mathbb{C}$$

satisfying certain transformation properties with respect to a matrix group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \right\} \subset SL_2(\mathbb{Z}).$$

Because of these properties, we have **Fourier expansions**:

$$f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z}.$$

We can also associate L-functions to such a form:

$$L(f, s) = \int_0^{\infty} f(it)\, t^s \, \frac{dt}{t} = \sum_{n \geq 1} \frac{a_n}{n^s}$$

Given an eigenform f with rational eigenvalues, we can build an elliptic curve E_f such that

$$L(E_f, s) = L(f, s)$$

defined over $\mathbb{Q}$

<u>Theorem</u>: For every elliptic curve E, there is some modular form f_E such that $L(E, s) = L(f_E, s)$.

with rational eigenvalues

1.2. Paramodularity

Let $\mathcal{H}_2$ be the following set:

$$\mathcal{H}_2 = \left\{ \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in M_{2\times 2}(\mathbb{C}) \;\middle|\; \begin{pmatrix} \operatorname{Im}\tau & \operatorname{Im} z \\ \operatorname{Im} z & \operatorname{Im}\omega \end{pmatrix} \text{ is positive definite} \right\}$$

A Siegel modular form is a holomorphic function $f : \mathcal{H}_2 \longrightarrow \mathbb{C}$ satisfying certain transformation properties with respect to a group of matrices $\Gamma \subset Sp_4(\mathbb{Z})$.

If the group Γ is the paramodular group

$$K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap Sp_4(\mathbb{Z}), \quad * \in \mathbb{Z},$$

then f is called a Siegel paramodular form.

As for modular forms, there are Hecke operators, eigenforms, and L-functions.

We want to use paramodular forms to generalise modularity.

<u>Definition</u>: An abelian variety is an algebraic variety A defined over k, with two morphisms

$$m: A \times A \longrightarrow A$$
$$inv: A \longrightarrow A$$

and $e \in A(k)$, such that $A(\bar{k})$ is a (commutative) group with (m, inv, e).

<u>Examples</u>:

1) Elliptic curves, $y^2 = x^3 + Ax + B$

2) Given $T \in \mathcal{H}_2$ $(T^t = T,\ \mathrm{Im}(T) > 0)$,

$$\mathbb{C}^2 / (\mathbb{Z}^2 \oplus \mathbb{Z}^2 T) \text{ is an abelian surface.}$$

Paramodular Conjecture (I)

There is a bijection

$$\left\{ \begin{array}{c} f \text{ paramodular} \\ \text{form with rational} \\ \text{eigenvalues} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{abelian surface} \\ A \text{ with trivial} \\ \text{endomorphism ring} \end{array} \right\}$$

such that $L(f,s) = L(A_f, s)$.

<u>Problem</u>: we do not have construction in either way!

2. Abelian varieties of GL_4-type

$$T_\ell A := \varprojlim A[\ell^n] \rightsquigarrow \rho_\ell : G_k \to GL_4(\mathbb{Q}_\ell)$$

2.1. Representations from abelian varieties

Let A be an abelian surface. For any prime ℓ we have a continuous representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_4(\mathbb{Q}_\ell)$$

which comes from the action of the Galois group on the points of A:

$$P = (P_0 : P_1 : \ldots : P_m) \longmapsto \sigma P = (\sigma(P_0) : \ldots : \sigma(P_m)).$$

The L-function of A is defined in terms of ρ_ℓ:

$$L(A,s) \text{ ``$=$'' } \prod_p \det\left(\rho_\ell(\mathrm{Frob}_p) - p^{-s}\,\mathrm{Id}\right)$$

Hence the connection Abelian varieties $\longleftrightarrow$ Paramodular forms requires studying the representations ρ_ℓ.

In general, if A has dimension n then we have a representation

$$\rho_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_{2n}(\mathbb{Q}_\ell)$$

for each $\ell \geq 2$ prime. **But** sometimes, we can lower the dimension of this representation.

$\to$ Need to study endomorphisms of A.

2.2. Endomorphism ring of A

Given an abelian variety A, its endomorphism ring is the set

$$\mathrm{End}(A) := \left\{ \varphi : A \to A \;\middle|\; \begin{array}{l} \varphi \text{ is a morphism of algebraic varieties, inducing a group homomorphism} \\ \varphi : A(\bar{\mathbb{Q}}) \to A(\bar{\mathbb{Q}}) \end{array} \right\}$$

It is a ring with sum and composition.

Technical (but important) points:

1) We prefer working with $\mathrm{End}(A) \otimes \mathbb{Q}$

2) Given a field $K \supset \mathbb{Q}$, we distinguish the set of endomorphisms defined over K by writing $\mathrm{End}(A_K)$.

For all integers $n \in \mathbb{Z}$, we have an endomorphism $[n]: A \to A$, defined by

$$P \longmapsto \begin{cases} P + \overset{n}{\cdots} + P, & \text{if } n > 0 \\ O, & \text{if } n = 0 \\ -(P + \overset{|n|}{\cdots} + P), & \text{if } n < 0. \end{cases}$$

If $m \neq n$, we have $[m] \neq [n]$. Hence

$\mathbb{Z} \subset \mathrm{End}(A) \Rightarrow \mathrm{End}(A)$ is a ring of characteristic zero.

Theorem: For all fields $\overline{\mathbb{Q}} \supset K \supset \mathbb{Q}$, $\mathrm{End}(A_K) \otimes \mathbb{Q}$ is finite-dimensional $\mathbb{Q}$-algebra

Examples

1) The elliptic curve $E: y^2 = x^3 + 1$ has

$$\mathrm{End}(E_{\mathbb{Q}}) \otimes \mathbb{Q} = \mathbb{Q}, \quad \mathrm{End}(E_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-3})$$

2) The curve $C: y^2 = x^5 + 1$ has an associated abelian surface $A = \mathrm{Jac}(C)$ with

$$\mathrm{End}(A_{\mathbb{Q}}) \otimes \mathbb{Q} = \mathbb{Q},$$

$$\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} = \mathbb{Q}(e^{2\pi i/5})$$

3) The curve $C: y^2 = x^5 - 2i\sqrt{2}\,x^4 - \frac{11}{3}x^3 + 2i\sqrt{2}\,x^2 + x$ has an associated surface $A = \mathrm{Jac}(C)$, such that

$$\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} = \left(\frac{3\,,\,-1}{\mathbb{Q}}\right) = \langle 1, i, j, ij \rangle_{\mathbb{Q}}$$

the quaternion algebra such that $i^2 = 3$, $j^2 = -1$.

2.3. Abelian varieties of GL_4-type

Note that "most" abelian surfaces A have $\mathrm{End}(A_{\bar{\mathbb{Q}}}) = \mathbb{Z}$. The degree of $\mathbb{Q}$ is 1, half of $\dim A = 2$.

Definition. An abelian variety A is of GL_4-type if $\mathrm{End}(A_{\mathbb{Q}})$ contains a number field of degree $[E:\mathbb{Q}] = \frac{1}{2}\dim A$.

Proposition. For each prime $\lambda \subset E$, there is a representation

$$\rho_\lambda : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_4(E_\lambda)$$

*(E_λ is a finite extension of $\mathbb{Q}_\ell$).

A general principle for studying Galois representations is that a representation

$$\rho_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(V)$$

is determined by its traces.

Define two fields:

$$H := \mathrm{Center}(\mathrm{End}(A_{\mathbb{Q}}))$$

$$F := \mathrm{Center}(\mathrm{End}(A_{\overline{\mathbb{Q}}}))$$

We have inclusions $F \subset H \subset E$.

We say a field F is totally real if, for all injections $\sigma : F \hookrightarrow \mathbb{C}$, we have

$$\sigma(F) \subset \mathbb{R}.$$

We suppose from now on that $F = Z(\mathrm{End}(A_{\bar{\mathbb{Q}}}))$ is a totally real field.

Theorem. If $E \subset \mathrm{End}(A_{\bar{\mathbb{Q}}})^{\otimes \mathbb{Q}}$ is a maximal field, then
$$\mathrm{End}(A_{\bar{\mathbb{Q}}}) \otimes \mathbb{Q} = M_n(D),$$
where D is either F, or a quaternion algebra defined over F.

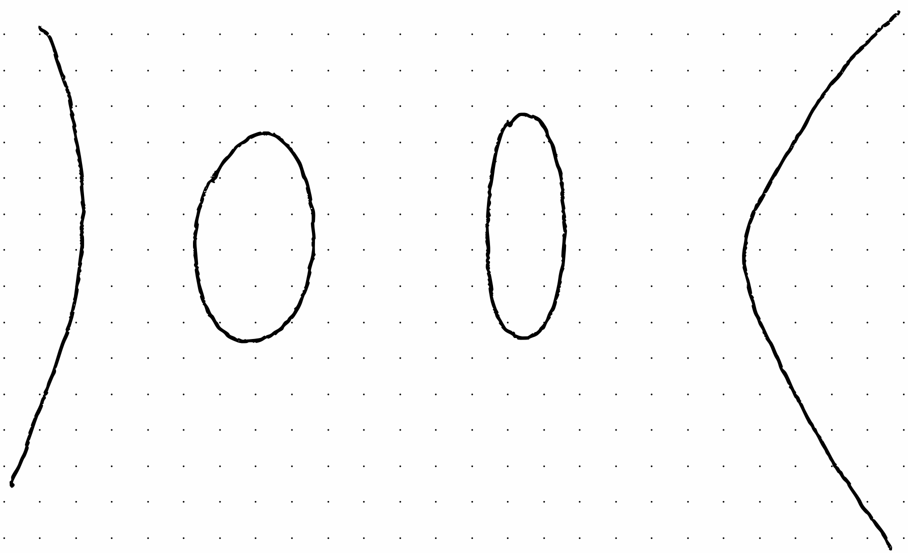
Geometrically, A is isogenous (over $\bar{\mathbb{Q}}$) to B^n, where B is a simple abelian variety over $\bar{\mathbb{Q}}$, and $\mathrm{End}(B_{\bar{\mathbb{Q}}}) \otimes \mathbb{Q} = D$

Proposition. H is the extension of F generated by the traces of ρ_d,

$$H = F(\{\mathrm{tr}\,\rho_d(\mathrm{Frob}_p)\}).$$

Proposition. The extension H/F is abelian. We call $\mathrm{Gal}(H/F)$ the group of inner twists of A.

3. Examples



3. Examples

Mestre's family of genus 4 curves

Let $K = \mathbb{Q}(v, a_1, a_2)$, and let C be the genus-4 curve

$$C : y^2 = (x - v)(vx - 1)(x^2 - a_1)(x^2 - a_2)\left(x^2 - \frac{a_1 v^2 - 1}{a_1 v^2}\right)\left(x^2 - \frac{a_2 v^2 - 1}{a_2 v^2}\right)$$

The fourfold $\mathrm{Jac}(C)$ is generically simple, and

$$\mathbb{Q}(\sqrt{2}) \subset \mathrm{End}(\mathrm{Jac}(C)_K).$$

A family of fourfolds with $F \subsetneq H$

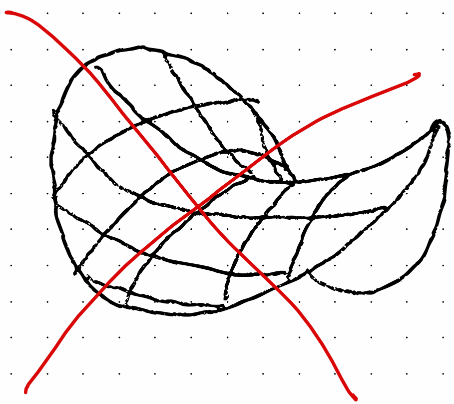
Let $r, s, t \in \mathbb{Q}$, $\quad w = \dfrac{r^2+2s^2+1}{2}$;

$$\begin{cases} F_1(x) = x^2 + (r+s\sqrt{-2})x + (w+t\sqrt{-2}) \\ F_2(x) = \sqrt{-2}\,x + (s - r\sqrt{-2}) \\ F_3(x) = -\sqrt{-2}\,x^2 - (-2s + r\sqrt{-2})x + \left(2t + \left(\tfrac{1}{2} - w\right)\sqrt{-2}\right). \end{cases}$$

Let $C: y^2 = F_1(x)F_2(x)F_3(x)$. If C is smooth, then $A = \mathrm{Res}_{\mathbb{Q}(\sqrt{-2})/\mathbb{Q}}(\mathrm{Jac}(C))$ is an abelian fourfold with

$$\underbrace{\mathrm{End}(A_{\mathbb{Q}}) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{2})}_{E = H} \quad \text{and} \quad \underbrace{\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} = M_2(\mathbb{Q})}_{F = \mathbb{Q}}$$

4. Fake surfaces



4. Fake surfaces

A fake abelian surface A is an abelian variety:

- of dimension 4
- such that $\mathrm{End}(A_{\mathbb{Q}})^{\otimes \mathbb{Q}} = \mathrm{End}(A_{\bar{\mathbb{Q}}})^{\otimes \mathbb{Q}}$ is a quaternion algebra over $\mathbb{Q}$, ie

$$D = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij$$

with $i^2 = a$, $j^2 = b$, $ij = -ji$; $a, b \in \mathbb{Q}^\times$.

Such a variety is of GL_4-type.

Since $\mathrm{End}(A_{\mathbb{Q}}) = \mathrm{End}(A_{\overline{\mathbb{Q}}})$, we have

$$F = H = \mathbb{Q}.$$

Hence all traces of ρ_ℓ lie in $\mathbb{Q}$.

Even more, for all $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$\det(\rho_\ell(g) - x \cdot \mathrm{Id}) = p(x)^2,$$

where $p(x)$ has degree four.

It's like we see double

(or twice the representation of a surface).

Calegari: if f is a Siegel paramodular form, it could happen that there is a fake surface A such that

$$L(f,s)^2 = L(A,s).$$

Hence, the problem with fake surfaces is that we cannot distinguish them from actual surfaces, at least not by looking at the traces of a representation.

Paramodular Conjecture (II)

There is abijection

$$\left\{\begin{array}{l}\text{Paramodular}\\ \text{eigenforms, w/}\\ \text{eigenvalues in } \mathbb{Q}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{abelian}\\ \text{surfaces}\\ \text{over } \mathbb{Q}\end{array}\right\} \cup \left\{\begin{array}{l}\text{fake}\\ \text{surfaces}\\ \text{over } \mathbb{Q}\end{array}\right\},$$

such that

$$L(A,s) = \begin{cases} L(f_A,s) & \text{if } \dim A = 2 \\ L(f_A,s)^2 & \text{if } \dim A = 4 \end{cases}$$