Abelian varieties that split modulo all but finitely many primes

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A local-global problem

Let k be a number field, A/k an abelian variety with $A_{\bar{k}}$ simple,
$\Sigma_A = \{\text{primes of good reduction for } A\}$.
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- Alternatively: to what extent does A "look like" a product $A_1 \times A_2$?

Known results

Theorem (Achter, Zywina)

Assume the Mumford-Tate conjecture for A. Suppose $\mathrm{End}(A_{\bar{k}})$ is **commutative**. There exists a finite extension k'/k such that, for a density one set of primes $\mathfrak{p} \in \Sigma_{A_{k'}}$, $A_{k',\mathfrak{p}}$ is simple.

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The main result

Theorem (F.)

Suppose $\mathrm{End}(A)$ is noncommutative. Then, for every prime $\mathfrak{p}$ of k of good reduction for A coprime to all primes of ramification of $\mathrm{End}(A) \otimes \mathbb{Q}$, the reduction $A_{\mathfrak{p}}$ splits.

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In particular, $S = \{\mathfrak{p} \in \Sigma_A \mid A_\mathfrak{p} \text{ is simple}\}$ is finite.

- Proof is guided by case of dim 2.
- Theorem applies to the following types in Albert classification:
  - Type II: indefinite quaternion algebra over totally real field.
  - Type III: definite quaternion algebra over totally real field.
  - Type IV: central division algebra over CM field.

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Let $q = p^r$. Consider a simple abelian variety $X/\mathbb{F}_q$ of dimension $g \geq 2$. Let $\mathrm{End}^0(X) = \mathrm{End}(X) \otimes \mathbb{Q}$.

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- For all $\mathfrak{q} \nmid p$, $\mathrm{inv}_{\mathfrak{q}}[\mathrm{End}^0(X)] = 0$.
- If $\mathbb{Q}(\pi)$ is CM, then for some $\mathfrak{p} \mid p$, $\mathrm{inv}_{\mathfrak{p}}[\mathrm{End}^0(X)] \neq 0$.

Proof of 2-dimensional case

Theorem (Morita, Yoshida)

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- Hence $\mathbb{Q}(\pi) =$ imaginary quadratic, and $\mathrm{End}^0(A_{\mathfrak{p}})$ ramifies at some prime over p.

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- $\Longrightarrow$ D ramifies at p. □

Problems when going to dimension > 2

- Need replacement for injection $\mathrm{End}^0(A) \otimes_{\mathbb{Q}} \mathbb{Q}(\pi) \hookrightarrow \mathrm{End}^0(A_{\mathfrak{p}})$.
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- Still need to compare $\mathrm{End}^0(A)$ and $\mathrm{End}^0(A_{\mathfrak{p}})$.
- If $Z = Z(\mathrm{End}^0(A))$ is CM field, then $\mathrm{ord}_{\mathrm{Br}(Z)}[\mathrm{End}^0(A)]$ can be arbitrarily large.

Reduction to prime subalgebras

Lemma

Let E be a division algebra with center Z and $\mathrm{ord}_{\mathrm{Br}(Z)}[E] = m$.
Let $\ell \mid m$. Then, there exist

- a field F with $Z \subseteq F \subset E$, and
- a central division F-algebra $D \subset E$ and $\mathrm{ord}_{\mathrm{Br}(F)}[D] = \ell$,

such that $[E \otimes_Z F] = [D]$ in $\mathrm{Br}(F)$.

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Remark

If $\mathrm{End}^0(A)$ is a quaternion algebra, then $D = \mathrm{End}^0(A)$.

Comparison of Brauer classes

Theorem (F.)

Let

- $B_{/K}$ a division algebra, $F \subset B$ a field.
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The proof uses a characterization of Chia-Fu Yu of embeddings $D \hookrightarrow B$.

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- Let $\mathfrak{p} \in \Sigma_A$ over p, suppose $A_{\mathfrak{p}}$ simple.

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- By previous result, letting $d = \mathrm{ord}_{\mathrm{Br}(F(\pi))}[\mathrm{End}^0(A_{\mathfrak{p}}) \otimes_{\mathbb{Q}(\pi)} F(\pi)]$, $\ell \| d$,

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- $\implies$ $\mathrm{End}^0(A)$ ramifies at a place over p. $\square$

Abelian varieties *with noncommutative endomorphism ring* split modulo all but finitely many primes

Enric Florit

Universitat de Barcelona

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